47) $$\int_0^1 \frac{1}{t+\sqrt{1+t^2}}\,dt = \int_0^{\alpha} \frac{1}{\operatorname{sh}x+\sqrt{1+\operatorname{sh}^2 x}}\operatorname{ch}x\,dx$$

CV
$t = \operatorname{sh} x$
« $dt = \operatorname{ch}x\,dx$ »
$t=0 : x=0$
$t=1 : x=\alpha$

$$= \int_0^{\alpha} \frac{\operatorname{ch}x}{\operatorname{sh}x+\sqrt{\operatorname{ch}^2 x}}\,dx$$

$$= \int_0^{\alpha} \frac{\operatorname{ch}x}{\operatorname{sh}x+\underbrace{|\operatorname{ch}x|}_{\geq 0}}\,dx$$

$$= \int_0^{\alpha} \frac{\operatorname{ch}x}{\operatorname{sh}x+\operatorname{ch}x}\,dx$$

$$= \int_0^{\alpha} \frac{\frac{e^x+e^{-x}}{2}}{e^x}\,dx$$

$$= \frac{1}{2}\int_0^{\alpha} \frac{e^x+e^{-x}}{e^x}\,dx$$

$$= \frac{1}{2}\int_0^{\alpha} \left(1+\frac{e^{-x}}{e^x}\right)dx$$

$$= \frac{1}{2}\int_0^{\alpha} \left(1+e^{-2x}\right)dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2}e^{-2x}\right]_0^{\alpha}$$

$$= \frac{1}{2}\left(\left(\alpha - \frac{1}{2}e^{-2\alpha}\right) - \left(0 - \frac{1}{2}e^{0}\right)\right)$$

$$= \frac{1}{2}\left(\alpha - \frac{1}{2}e^{-2\alpha} + \frac{1}{2}\right)$$

$$= \frac{1}{2}\alpha - \frac{1}{4}e^{-2\alpha} + \frac{1}{4}$$

Or $\mathrm{ch}^2\alpha = 1 + \underbrace{\mathrm{sh}^2\alpha}_{=1} = 2$ donc $\underbrace{\mathrm{ch}\,\alpha}_{\geqslant 0} = \cancel{\pm}\sqrt{2}$

donc, $e^{\alpha} = \mathrm{ch}\,\alpha + \mathrm{sh}\,\alpha = 1 + \sqrt{2}$ d'où $\alpha = \ln(1+\sqrt{2})$

et $e^{2\alpha} = (1+\sqrt{2})^2 = 3 + 2\sqrt{2}$

Ainsi, $\frac{1}{2}\alpha - \frac{1}{4}e^{-2\alpha} + \frac{1}{4} = \frac{1}{2}\ln(1+\sqrt{2}) - \frac{1}{4} \times \frac{1}{3+2\sqrt{2}} + \frac{1}{4}$

$$= \frac{1}{2}\ln(1+\sqrt{2}) - \frac{1}{4(3+2\sqrt{2})} + \frac{3+2\sqrt{2}}{4(3+2\sqrt{2})}$$

$$= \frac{1}{2}\ln(1+\sqrt{2}) + \frac{2+2\sqrt{2}}{4(3+2\sqrt{2})}$$

$$= \frac{1}{2}\ln(1+\sqrt{2}) + \frac{1+\sqrt{2}}{2(3+2\sqrt{2})}$$

$$= \frac{1}{2}\left(\ln(1+\sqrt{2}) + \frac{(1+\sqrt{2})(3-2\sqrt{2})}{\underbrace{(3+2\sqrt{2})(3-2\sqrt{2})}_{=1}}\right)$$

$$= \frac{1}{2}\left(\ln(1+\sqrt{2}) + 3 - 2\sqrt{2} + 3\sqrt{2} - 4\right)$$

$$= \frac{1}{2}\left(\ln(1+\sqrt{2}) + \sqrt{2} - 1\right)$$

donc $\displaystyle\int_0^1 \frac{dt}{t+\sqrt{1+t^2}} = \boxed{\frac{\sqrt{2}-1+\ln(1+\sqrt{2})}{2}}$

(Concluez proprement !)