19 Soient p, q deux projecteurs sur un espace vectoriel E tel que $p \circ q = 0$. Montrer que $r = p + q - q \circ p$ est un projecteur sur $\operatorname{Im} p + \operatorname{Im} q$ parallèlement à $\operatorname{Ker} p \cap \operatorname{Ker} q$.

19 p, q projection sur E

tel que $p \circ q = 0_{\mathcal{L}(E)}$

$[p, q \in \mathcal{L}(E), \quad p^2 = p, \quad q^2 = q]$

$r = p + q - q \circ p$

<u>But</u> : r projection

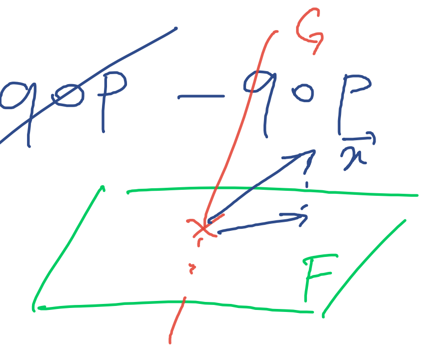
$r \in \mathcal{L}(E)$ par opérations.

$$r^2 = (p+q-q\circ p)^2$$
$$= (p+q-q\circ p)\circ(p+q-q\circ p)$$

$$= p^{\cancel{2}} + \cancel{p\circ q} - \cancel{(p\circ q)\circ p}$$
$$+ q\circ p + q^{\cancel{2}} - q^{\cancel{2}}\circ p$$
$$- q\circ p^{\cancel{2}} - \cancel{q\circ(p\circ q)} - \cancel{q\circ(p\circ q)\circ p}$$

$$= p + \cancel{q\circ p} + q - \cancel{q\circ p} - q\circ p$$
$$= r$$

une projection



r projection sur $\mathrm{Im}\, r = \mathrm{Inv}\, r$ parallèlement $\mathrm{Ker}\, r$.

* <u>But</u> : $\mathrm{Ker}\, r = \mathrm{Ker}\, p \cap \mathrm{Ker}\, q$

$\supset$ Si $x \in \mathrm{Ker}\, p \cap \mathrm{Ker}\, q$,

$p(x) = q(x) = 0_E$ donc

$r(x) = p(x) + q(x) - p(q(x)) = 0_E$

Si $x \in \operatorname{Ker} r$ ie $r(x) = 0_E$

But : $p(x) = q(x) = 0_E$.

Or $p(x) + q(x) - q \circ p(x) = 0_E$

En composant par p,

$\cancel{p^2(x) + p \circ q(x) - (p \circ q) \circ p(x)}$

$= p(0_E) = 0_E$

donc $p(x) = 0_E$.

En composant par q,

$\cancel{q \circ p(x) + q^2(x) - q^2 \circ p(x)}$

$= q(0_E) = 0_E$

donc $q(x) = 0_E$.

Donc $\boxed{\operatorname{Ker} r = \operatorname{Ker} p \cap \operatorname{Ker} q.}$

* $\operatorname{Im} r = \operatorname{Im} r = \operatorname{Im} p + \operatorname{Im} q$

⊂ Si $x \in E$,

$r(x) = p(x) + q(x) - q \circ p(x)$

$= \underbrace{p(x)}_{\in \operatorname{Im} p} + \underbrace{q(x - p(x))}_{\in \operatorname{Im} q}$

⊃ Si $y \in \operatorname{Im} p + \operatorname{Im} q$, on a $x, x' \in E$ tels que $y = p(x) + q(x')$.

$$r(y) = r(p(x) + q(x'))$$

$$= r(p(x)) + r(q(x'))$$

$$= p^{\cancel{2}}(x) + \cancel{q\circ p(x)} - \cancel{q\circ p^2(x)}$$

$$+ \cancel{p\circ q(x')} + q^{\cancel{2}}(x') - \cancel{q\circ p\circ q(x')}$$

$$= p(x) + q(x') = y$$

donc $y \in \mathrm{Inv}\, r = \mathrm{Im}\, r$.