26 $u \in \mathcal{L}(E)$ avec E de df.

1) $\operatorname{Ker} u \oplus \operatorname{Im} u = E$ ie $\operatorname{Ker} u \cap \operatorname{Im} u = \{0_E\}$

2) $\operatorname{Ker} u = \operatorname{Ker} u^2$ ie $\operatorname{Ker} u^2 \subset \operatorname{Ker} u$

3) $\operatorname{Im} u = \operatorname{Im} u^2$ ie $\operatorname{Im} u \subset \operatorname{Im} u^2$

$$\boxed{\begin{array}{ccc} 1) & \Rightarrow & 2) \\ \Uparrow & & \Downarrow \\ & 3) & \end{array}}$$

__1) ⇒ 2)__ : Si $\operatorname{Ker} u \oplus \operatorname{Im} u = E$
ie $\operatorname{Ker} u \cap \operatorname{Im} u = \{0_E\}$

__But__ : $\operatorname{Ker} u = \operatorname{Ker} u^2$

On a déjà $\operatorname{Ker} u \subset \operatorname{Ker} u^2$

__But__ : $\operatorname{Ker} u^2 \subset \operatorname{Ker} u$

Soit $x \in \operatorname{Ker} u^2$: $u^2(x) = 0_E$.

__But__ : $u(x) = 0_E$.

Or $$u\big(\underbrace{u(x)}_{\substack{\in \operatorname{Ker} u \cap \operatorname{Im} u \\ = \{0_E\}}}\big) = 0_E$$

donc $x \in \operatorname{Ker} u$.

__2) ⇒ 3)__ : Si $\operatorname{Ker} u = \operatorname{Ker} u^2$

__But__ : $\operatorname{Im} u = \operatorname{Im} u^2$

On a déjà $\operatorname{Im} u^2 \subset \operatorname{Im} u$

__But__ : $\operatorname{Im} u = \operatorname{Im} u^2$

$$\begin{aligned}\dim(\operatorname{Im} u) &= \operatorname{rg} u = \dim E - \dim \operatorname{Ker} u \\ &= \dim E - \dim \operatorname{Ker} u^2 = \operatorname{rg} u^2 \\ &= \dim(\operatorname{Im} u^2)\end{aligned}$$

donc $\mathrm{Im}\, u = \mathrm{Im}\, u^2$.

$\cancel{3}^{\,2}) \Longrightarrow 1)$: Si $\cancel{\mathrm{Im}}^{\,\mathrm{Ker}}\, u = \cancel{\mathrm{Im}}^{\,\mathrm{Ker}}\, u^2$,

But : $\mathrm{Ker}\, u \oplus \mathrm{Im}\, u = E$

On a déjà $\dim \mathrm{Ker}\, u + \dim \mathrm{Im}\, u = \dim E$.

But : $\mathrm{Ker}\, u \cap \mathrm{Im}\, u = \{0_E\}$.

Si $y \in \mathrm{Ker}\, u \cap \mathrm{Im}\, u$,

On a $x \in E$ tel que

$$\begin{cases} y = u(x) \\ u(y) = 0_E \end{cases}$$

donc $u^2(x) = 0_E$.

donc $y = u(x) = 0_E$ car $\mathrm{Ker}\, u = \mathrm{Ker}\, u^2$

$3) \Longrightarrow 1)$: Si $\mathrm{Im}\, u = \mathrm{Im}\, u^2$,

But : $\mathrm{Ker}\, u + \mathrm{Im}\, u = E$

Soit $x \in E$

On cherche $x' \in \mathrm{Ker}\, u$, $x'' \in E$ tq

$$x = \underbrace{x'}_{\in \mathrm{Ker}\, u} + \underbrace{u(x'')}_{\in \mathrm{Im}\, u} \quad \text{donc } u(x) = u^2(x'')$$

$u(x) \in \mathrm{Im}\, u = \mathrm{Im}\, u^2$

On a $x'' \in E$ tel que

$$u(x) = u^2(x'') \text{ donc } u(x - u(x'')) = 0_E$$

Soit $x' = x - u(x'') \in \mathrm{Ker}\, u$.

donc $x = x' + u(x'') \in \mathrm{Ker}\, u + \mathrm{Im}\, u$.