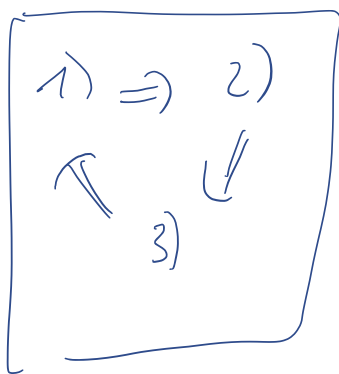


26) $u \in \mathcal{L}(E)$ avec E de df.

1) $\text{Ker } u \oplus \text{Im } u = E$ ie $\text{Ker } u \cap \text{Im } u = \{0_E\}$

2) $\text{Ker } u = \text{Ker } u^2$ ie $\text{Ker } u^2 \subset \text{Ker } u$

3) $\text{Im } u = \text{Im } u^2$ ie $\text{Im } u \subset \text{Im } u^2$



1) \Rightarrow 2) : Si $\text{Ker } u \oplus \text{Im } u = E$
ie $\text{Ker } u \cap \text{Im } u = \{0_E\}$

But : $\text{Ker } u = \text{Ker } u^2$

On a déjà $\text{Ker } u \subset \text{Ker } u^2$

But : $\text{Ker } u^2 \subset \text{Ker } u$

Soit $x \in \text{Ker } u^2$: $u^2(x) = 0_E$.

But : $u(x) = 0_E$.

Or $u(\underbrace{u(x)}) = 0_E$
 $\in \text{Ker } u \cap \text{Im } u$
 $= \{0_E\}$

donc $x \in \text{Ker } u$.

2) \Rightarrow 3) : Si $\text{Ker } u = \text{Ker } u^2$

But : $\text{Im } u = \text{Im } u^2$

On a déjà $\text{Im } u^2 \subset \text{Im } u$

But : $\text{Im } u = \text{Im } u^2$

$\dim(\text{Im } u) = \text{rg } u = \dim E - \dim \text{Ker } u$
 $= \dim E - \dim \text{Ker } u^2 = \text{rg } u^2$
 $= \dim(\text{Im } u^2)$

donc $\text{Im } u = \text{Im } u^2$.

2) \Rightarrow 1) : Si $\overset{\text{ker}}{\text{Im}} u = \overset{\text{ker}}{\text{Im}} u^2$,

But : $\text{Ker } u \oplus \text{Im } u = E$

On a déjà $\dim \text{Ker } u + \dim \text{Im } u = \dim E$.

But : $\text{Ker } u \cap \text{Im } u = \{0_E\}$.

Si $y \in \text{Ker } u \cap \text{Im } u$,

On a $x \in E$ tel que

$$\begin{cases} y = u(x) \\ u(y) = 0_E \end{cases}$$

donc $u^2(x) = 0_E$.

donc $y = u(x) = 0_E$ car $\text{Ker } u = \text{Ker } u^2$

3 \Rightarrow 1) : Si $\text{Im } u = \text{Im } u^2$,

But : $\text{Ker } u + \text{Im } u = E$

Soit $x \in E$

[On cherche $x' \in \text{Ker } u, x'' \in E$ tq
 $x = \underbrace{x'}_{\in \text{Ker } u} + \underbrace{u(x'')}_{\in \text{Im } u}$ donc $u(x) = u^2(x'')$]

$u(x) \in \text{Im } u = \text{Im } u^2$

On a $x'' \in E$ tel que

$u(x) = u^2(x'')$ donc $u(x - u(x'')) = 0_E$

Soit $x' = x - u(x'') \in \text{Ker } u$.

donc $x = x' + u(x'') \in \text{Ker } u + \text{Im } u$.